Section 3.5

reminder: HW7 is currently active → due this Fri
HW8 will drop this Sat AM

previously, saw how to solve HG linear eq'ns w constant coefficients

recall: gen soln of non-HG nth order linear eqns has form:

$$Y = y_c + y_p$$

where y_c is complementary soln, which is gen. soln associated to associated HG eqn:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0$$

and y_p is particular sol'n of non-HG eq'n:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

suppose $f(x)$ is a polynomial with a degree m

$\mathbb{P}$

try to determine y_p — know $\mathbb{R}$ = reals

know derivatives of $\mathbb{P}$ are $\mathbb{P}$'s of a lower degree

for now, set up y_p as $\mathbb{P}$ with degree m w/undetermined coefficients

$$y_p = A_m x^m + A_{m-1} x^{m-1} + \ldots + A_1 x + A_0$$

could easily make it a sol'n of the non-HG eqn by equating coefficients of like powers of x

ex. find y_p given $y'' + 3y' + 4y = 3x + 2$ ← $f(x)$ has a degree of 1

guess $y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

substitute: $0 + 3A + 4(Ax + B) = 3x + 2$

$$3A + 4Ax + 4B = 3x + 2$$

$4A = 3$
$A = \frac{3}{4}$

$3A + 4B = 2$
$3\frac{3}{4} + 4B = 2$
$\frac{9}{4} + 4B = 2 = \frac{8}{4}$
$4B = -\frac{1}{4} \Rightarrow B = -\frac{1}{16}$

$\therefore \boxed{y_p = \frac{3}{4}x - \frac{1}{16}}$

now suppose $f(x) = a\cos kx + b\sin kx$

then $y_p = A\cos kx + B\sin kx$ since any deriv of LC of $\cos kx$ and $\sin kx$ would have the same form

ex. find y_p given $3y'' + y' - 2y = 2\cos x$

might guess $y_p = A\cos x$

however, need a term w $\sin x$ also

instead, choose $y_p = A\cos x + B\sin x$
$y_p' = -A\sin x + B\cos x$
$y_p'' = -A\cos x - B\sin x$

substitute:
$$3(-A\cos x - B\sin x) - A\sin x + B\cos x - 2(A\cos x + B\sin x) = 2\cos x$$
$$-3A\cos x - 3B\sin x - A\sin x + B\cos x - 2A\cos x - 2B\sin x = 2\cos x$$

$-3A + B - 2A = 2$ $\quad$ $-3B - A - 2B = 0$ $\quad \Rightarrow A = -\frac{5}{13}, \ B = \frac{1}{13}$

(solve on your own later w method of your choice)

$\therefore \boxed{y_p = -\frac{5}{13}\cos x + \frac{1}{13}\sin x}$

ex. find y_p given $y'' - 4y = 2e^{3x}$

$f(x) = 2e^{3x}$

$\vdots$

$f^{(n)}(x) = \underbrace{2(3^n)}_{\text{constant/multiple}} e^{3x}$

$y_p = Ae^{3x}$

substitution: $9Ae^{3x} - 4Ae^{3x} = 2e^{3x}$

$9A - 4A = 2$

$5A = 2 \Rightarrow A = \frac{2}{5}$

$\therefore \boxed{y_p = \frac{2}{5}e^{3x}}$

next example is not as straightforward:

ex. find y_p given $y'' - 4y = 2e^{2x}$

guess $y_p = Ae^{2x}$ (logical choice)

$y_p'' \; 4Ae^{2x}$

$4Ae^{2x} - 4Ae^{2x} = 2e^{2x}$

$0 \neq 2e^{2x}$ ← can never be zero

instead try $y_p = Axe^{2x}$

then $y_p' = Ae^{2x} + 2Axe^{2x}$

$y_p'' = \underbrace{2Ae^{2x} + 2Ae^{2x}} + 4Axe^{2x}$

$= 4Ae^{2x} + 4Axe^{2x}$

subst. take two: $4Ae^{2x} + \cancel{4Axe^{2x}} - \cancel{4Axe^{2x}} = 2e^{2x}$

$4A = 2 \Rightarrow A = \frac{1}{2}$

$\therefore \boxed{y_p = \frac{1}{2}xe^{2x}}$

issue with previous example was that $f(x) = 2e^{2x}$ satisfied the associated HG eq'n not the non-HG eq'n

if there's no such conflict with trial y_p, use what's called <u>Method of Undetermined Coefficients</u> ← 2 rules

<u>Rule 1</u> Assuming no term in $f(x)$ or in any of its derivatives satisfies associated HG eq'n, take a trial y_p using a LC of all linearly independent terms and

a LC of all linearly independent terms <u>and</u> their derivatives.

Determine coefficients using substitution.

ex. find y_p of $y'' + 4y = 3x^3$ ← degree of 3

$$y_p = Ax^3 + Bx^2 + Cx + D$$
$$y_p' = 3Ax^2 + 2Bx + C$$
$$y_p'' = 6Ax + 2B$$

substitute:
$$6Ax + 2B + 4(Ax^3 + Bx^2 + Cx + D) = 3x^3$$

solve on your own later:

$A = \frac{3}{4}$ $B = 0$ $C = -\frac{9}{8}$ $D = 0$

$$\therefore \boxed{y_p = \frac{3}{4}x^3 - \frac{9}{8}x}$$

ex. Solve non-HG IVP given $y'' - 3y' + 2y = \underbrace{3e^{-x} - 10\cos 3x}$ $y(0) = 1$, $y'(0) = 2$

$Y = y_c + y_p$

first consider associated HG eqn

Char eqn becomes: $r^2 - 3r + 2 = 0$
$$(r-1)(r-2) = 0$$
$$r_1 = 1 \quad r_2 = 2$$
$$y_c = c_1 e^{1x} + c_2 e^{2x}$$

compare to RHS

neither term in y_c or their derivatives

try $y_p = Ae^{-x} + B\cos 3x + C\sin 3x$
$$y_p' = -Ae^{-x} - 3B\sin 3x + 3C\cos 3x$$
$$y_p'' = Ae^{-x} - 9B\cos 3x - 9C\sin 3x$$

try on your own later

$A = \frac{1}{2}$ $B = \frac{7}{13}$ $C = \frac{9}{13}$

$$y_p = \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$$

$Y = y_c + y_p = c_1 e^x + c_2 e^{2x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$

use initial cond. $y(0)=1$ $y'(0)=2$ to solve for c_1, c_2

$y(0) = c_1 + c_2 + \frac{1}{2} + \frac{7}{13} + 0 = 1$

$c_1 + c_2 = -\frac{1}{26}$

$y' = c_1 e^x + 2c_2 e^{2x} - \frac{1}{2}e^{-x} - 3\left(\frac{7}{13}\right)\sin 3x + 3\left(\frac{9}{13}\right)\cos 3x$

$y'(0) = c_1 + 2c_2 - \frac{1}{2} - 0 + \frac{27}{13} = 2$

$c_1 + 2c_2 = \frac{11}{26}$

solve to find $c_1 = -\frac{1}{2}$ $c_2 = \frac{6}{13}$

$\therefore$ desired soln is:

$$\boxed{y(x) = -\frac{1}{2}e^x + \frac{6}{13}e^{2x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x}$$

Previous technique does NOT always work

ex. find y_p given $y''' + y'' = 3e^x + 4x^2$

char eqn: $r^3 + r^2 = 0$

$r^2(r+1) = 0$

$r_1 = 0$ $r_2 = -1$

mult: 2

$y_c = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{-1x}$

$= c_1 + c_2 x + c_3 e^{-x}$

compare to $f(x)$ on RHS

problem: duplication in terms

start trial $y_p = Ae^x + B + Cx + Dx^2$

to "fix" multiply $B + Cx + Dx^2$ by x^2:

y_p becomes $Ae^x + Bx^2 + Cx^3 + Dx^4$

$y_p' = Ae^x + 2Bx + 3Cx^2 + 4Dx^3$

$y_p'' = Ae^x + 2B + 6Cx + 12Dx^2$

$y_p''' = Ae^x + 6C + 24Dx$

substitute to find $A = \frac{3}{2}$ $B = 4$ $C = -\frac{4}{3}$ $D = \frac{1}{3}$

$y_p = Ae \quad + 6C \quad + 24Dx$

substitute to find $A = \frac{3}{2}$ $B = 4$ $C = -\frac{4}{3}$ $D = \frac{1}{3}$

$\therefore \boxed{y_p = \frac{3}{2}e^x + 4x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4}$

ex. determine form of y_p given $y'' + 6y' + 13y = e^{-3x}\cos 2x$

Char eqn: $r^2 + 6r + 13 = 0$

$r = -3 \pm 2i$

$y_c = e^{-3x}(c_1\cos 2x + c_2 \sin 2x)$

trial y_p: $e^{-3x}(A\cos 2x + B\sin 2x) \xrightarrow{\text{adaptation}} y_p = e^{-3x}(Ax\cos 2x + Bx\sin 2x)$